TWO-DIMENSIONAL TRANSONIC GAS FLOW IN THE PRESENCE OF A NORMAL SHOCK WAVE

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Two kinds of transonic flows are considered in this paper, viz. the free sonic stream past a profile, and the flow in a symmetric Laval nozzle, both in the presence of a weak shock wave. The flows are analyzed in the $\varphi\psi$-plane. In the neighborhood of the sonic stream boundary, or of the nozzle axis, the shock wave assumed to be approximately normal. Exact nonself-similar solutions θ, η of the Falkovich nonlinear transonic equations corresponding to such flows are derived.

1. We shall consider two-dimensional transonic flows of a perfect gas in the presence of shock waves. With a weak shock the velocities in the stream do not differ appreciably from the critical velocity of sound, and if the stream vorticity is neglected, we find that Frankl's variables [1] θ, η satisfy the system of Eqs.

$$\frac{\partial\eta}{\partial\psi} = -B_0\frac{\partial\theta}{\partial\varphi}, \quad \frac{\partial\theta}{\partial\psi} = B_0\eta\frac{\partial\eta}{\partial\varphi}, \quad B_0 = (k+1)^{1/3}\left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}, \quad k = \frac{c_p}{c_v} \tag{1.1}$$

Here φ is the velocity potential, and ψ the stream function. We shall assume that the shock is normal, i.e. that all streamlines intersect the latter at a straight angle. Selecting the origin of the xy-coordinates at a certain characteristic point A of the shock wave, and the x-axis oriented in the direction of the flow velocity, we find that along a normal shock wave $\theta \equiv 0$, $\varphi \equiv 0$ [3].

Let $\varphi > 0$ upstream of the shock wave, and $\varphi < 0$ downstream of it, with $\psi = 0$ along the streamline passing through the characteristic point A ($x = y = 0$).

For a weak normal transonic shock wave the following conditions must be fulfilled [3]

$$\theta(0, \psi) = 0, \qquad \eta(+0, \psi) = -\eta(-0, \psi) \tag{1.2}$$

A solution of system (1.1) which would satisfy the first of conditions (1.2) may be sought in the form of a power series [4]

$$\theta = \sum_{n=0}^{\infty} z_n'(\psi)\frac{\varphi^{2n+1}}{2n+1}, \qquad \eta = -B_0\sum_{n=0}^{\infty} z_n(\psi)\varphi^{2n} \tag{1.3}$$

$$z_n''(\psi) = (2n+1)B_0^3\sum_{k=0}^{n}(2n-2k+2)z_k(\psi)z_{n-k+1}(\psi) \qquad (n = 0, 1, 2, \ldots)$$

It follows from the second of conditions (1.2) that $z_0(\psi) \not\equiv 0$. We shall derive the exact solution of system (1.1) by retaining the first two terms of expansion (1.3) (retention of the first term only would yield a simple self-similar solution of system (1.1)).

Then the solution upstream of the shock wave is of the form

$$\theta = [z_0'(\psi) + {}^1/_3\, z_1'(\psi)\varphi^2]\varphi, \qquad \eta = -B_0[z_0(\psi) + z_1(\psi)\varphi^2] \quad (\varphi \geqslant 0) \tag{1.4}$$

in which functions $z_0(\psi)$ and $z_1(\psi)$ satisfy Eqs.

$$z_1'' = 6B_0^3 z_1^2, \qquad z_0'' = 2B_0^3 z_1 z_0 \tag{1.5}$$

Downstream of the shock the solution of (1.1) is of the form (1.6)

$$\theta = [z_0'(\psi) - {}^1/_3\, z_1'(\psi)\,\varphi^2]\,(-\varphi), \qquad \eta = B_0\,[z_0(\psi) - z_1(\psi)\varphi^2] \quad (\varphi \leqslant 0)$$

It follows from (1.2) and (1.5) that functions z_0 and z_1 appearing in (1.4) and (1.6) are the same, while $z_0(\psi) \geqslant 0$.

The solution of system (1.1) in the form (1.4) was derived by Tomotika and Tamada [5] and was used for the analysis of a shockless transonic flow in a symmetric two-throated Laval nozzle. Here this solution is applied to flows in the presence of shock waves. We note that the assumption of $z_0 \equiv 0$ (no shock) yields the solution derived by Ovsiannikov [4] which describes a flow of gas with a normal sonic line.

The transposition of the derived solution from the $\varphi\psi$-plane into the xy flow plane is achieved with the aid of Formulas

$$dx = \frac{\cos\theta}{v}\,d\varphi - \frac{\rho_0}{\rho}\frac{\sin\theta}{v}\,d\psi, \qquad dy = \frac{\sin\theta}{v}\,d\varphi + \frac{\rho_0}{\rho}\frac{\cos\theta}{v}\,d\psi \tag{1.7}$$

while for the motion of a Trikomi gas subject to system (1.1) the following relationships hold [6]

$$\frac{\rho_0}{\rho v} = B_0\frac{d}{d\eta}\frac{1}{v}, \qquad \frac{d}{d\eta}\frac{\rho_0}{\rho v} = B_0\frac{\eta}{v}$$

It follows from this that function $1/v = f(\eta)$ satisfies the Airy equation [6 and 7]

$$f'' - \eta f = 0$$

$$a_* f(0) = 1, \quad a_* f'(0) = (k+1)^{-1/3}$$

where a_* is the critical velocity of sound at initial conditions.

Hence, v and ρ/ρ_0 may be expressed [6 and 7] by Airy's functions $\mathrm{Ai}(\eta)$ and $\mathrm{Bi}(\eta)$.

2. We shall consider a stream flowing past a profile in which a shock wave is generated. Let a plane sonic stream of a perfect gas emerging from a slit in a vessel wall flow past a profile (Fig. 1).

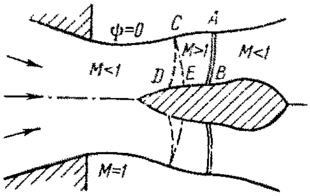

Fig. 1

At the stream free boundary the velocity is sonic ($M = 1$, $\eta = 0$), while upstream of the sonic line CD the flow in the stream is subsonic, and becomes supersonic downstream of that line.

The symmetric vortex-free flow of gas at some distance from a profile placed in a sufficiently wide sonic stream was analyzed in [8] up to the limiting characteristic CE, and it was shown in that paper that the flow in the neighborhood of point C conforms to the Barantsev's solution ([9] Chapter IV), therefore C is an inflection point of the stream boundary. Beyond that point the angle of inclination of the stream boundary to its axis decreases monotonously; the supersonic stream is slowed down by shock waves (the first of these is shown on Fig. 1 by line AB).

We shall investigate the flow in the neighborhood of the stream boundary CA, and shall assume the shock wave to be there normal. Let $\psi = 0$ along CA and $\psi > 0$ within the stream itself.

Conditions (1.2) must be supplemented by condition at the free sonic stream boundary

$$\eta(\varphi, 0) = 0 \tag{2.1}$$

Hence, from (1.4) and (1.6) we have

$$z_0(0) = 0, \quad z_1(0) = 0, \quad z_0(\psi) > 0 \qquad (\psi > 0) \tag{2.2}$$

From (1.5) and (2.2) follows

$$(z_1')^2 = 4B_0^3\, z_1^3 + [z_1'(0)]^2$$

and $z_1(\psi)$ may be expressed [10] in terms of the Weierstrass elliptic function $\wp(u;\, g_2, g_3)$ with parameters $g_2 = 0$, $g_3 = -4$, $e_2 = -1$

$$z_1(\psi) = N^2 B_0^{-3}\wp(u), \qquad \wp(u) = \wp(u;\, 0, -4)$$

$$z_1'(\psi) = (N/B_0)^3 \wp'(u), \qquad u = N\psi + \alpha \qquad (0 < u < 2\omega,\ N > 0) \qquad (2.3)$$

Function $\wp(u)$ satisfies Eq. $(\wp')^2 = 4\wp^3 + 4$. It follows from (2.2) that α is a zero of function $\wp(u)$.

Because $\Delta = g_2^3 - 27g_3^2 < 0$, $e_2 < 0$, therefore $\wp(u;\ 0, -4)$ has zeros (two) in the fundamental parallelogram for real values of u only, and the function itself is then real ([11] Section 86). Therefore N and α in (2.3) are real constants (by virtue of parity of $\wp(u)$ we consider $N > 0$), $0 < u < 2\omega$, where ω is the real half-period of $\wp(u)$ (see [11]):

$$\omega = \int_{-1}^{\infty} \frac{dx}{2\sqrt{1+x^3}} = \frac{\sqrt{3}}{2^{1/3}} \frac{\Gamma^3(1/3)}{4\pi} \approx 2.1033 \qquad (2.4)$$

In the fundamental interval $(0,\ 2\omega)$ function $\wp(u)$ is symmetric with respect to point $u = \omega$, at which $\wp(\omega) = -1$. Because $\wp(u) = 2^{2/3}\wp(x;\ 0, -1)$, $x = 2^{1/3}u$, we can make use of the graph of function $\wp(x;\ 0, -1)$ given in [5]. We denote the α-zeros of function $\wp(u)$ by α_1 and α_2 $(0 < \alpha_1 < \alpha_2 < 2\omega)$ and we obtain

$$\alpha_1 = \int_0^{\infty} \frac{dx}{2\sqrt{1+x^3}} \qquad (2.5)$$

Expressing integrals (2.4) and (2.5) in terms of gamma-functions [10], we obtain

$$\alpha_1 = {}^2/_3\ \omega, \qquad \alpha_2 = {}^4/_3\ \omega \qquad (2.6)$$

i.e. the zeros of $\wp(u)$ subdivide interval $(0,\ 2\omega)$ into three equal parts [4, 5 and 9], with $\wp'(\alpha_1) = -2$, $\wp'(\alpha_2) = 2$.

We adduce several particular values of this function

$$\wp\left(\frac{\omega}{3}\right) = 2, \qquad \wp\left(\frac{\omega}{2}\right) = \sqrt{3} - 1, \qquad \wp\left(\frac{2\omega}{3}\right) = 0, \qquad \wp(\omega) = -1$$

We shall now determine function $z_0(\psi)$. It follows from (1.5), (2.2), (2.3) that $z_0(\psi) = R(u)$ (where $u = N\psi + \alpha$) satisfies linear equation

$$R''(u) = 2\wp(u) R(u) \qquad (\alpha \leqslant u < 2\omega) \qquad (2.7)$$

with initial conditions

$$R(\alpha) = 0, \qquad R'(\alpha) > 0 \qquad (2.8)$$

It may be readily ascertained that the general solution of (2.7) is [5]

$$R(u) = A_1[\wp'(u) + 2]^{1/3} + A_2[\wp'(u) - 2]^{1/3} \qquad (2.9)$$

Hence two solutions corresponding to zeros α_1 and α_2 of function $\wp(u)$ which will satisfy conditions (2.2) and (2.8) can be obtained:

the first ($\alpha = \alpha_1$)

$$z_0(\psi) = R(u) = N^2 B_0^{-3} \varphi_c^2 [\wp'(u) + 2]^{1/3} \qquad (\alpha_1 \leqslant u < 2\omega)$$

$$z_0'(\psi) = NR'(u) = 2^{-1/3}(N/B_0)^3 |2 - \wp'(u)|^{2/3} \varphi_c^2 \qquad (u = N\psi + \alpha_1) \qquad (2.10)$$

the second ($\alpha = \alpha_2$)

$$z_0(\psi) = N^2 B_0^{-3} \varphi_c^2 [\wp'(u) - 2]^{1/3} \qquad (\alpha_2 \leqslant u < 2\omega)$$

$$z_0'(\psi) = 2^{-1/3}(N/B_0)^3 \varphi_c^2 [\wp'(u) + 2]^{2/3} \qquad (u = N\psi + \alpha_2) \qquad (2.11)$$

Here φ_c is a certain constant.

We shall derive the solution of (1.1) corresponding to the flow in the neighborhood of boundary CA shown on Fig. 1. In order to have all streamlines intersecting the sonic line $\eta = 0$ upstream of the shock wave, it is necessary to stipulate in (1.4) where $z_0(\psi) \geqslant 0$ that $z_1(\psi) \leqslant 0$, hence it follows from (2.3) that

$$\wp(u) \leqslant 0, \qquad \alpha = \alpha_1 \qquad (\alpha_1 \leqslant u < \alpha_2)$$

The solution of the system for such flow is of the form

$$\theta = (N/B_0)^3 [2^{-1/3}\varphi_c^2 |2 - \wp'(u)|^{2/3} |\varphi| + {}^1/_3 \varphi^3 \wp'(u)], \qquad (u = N\psi + \alpha_1) \qquad (2.12)$$

$$\eta = -(N/B_0)^2 \{\varphi_c^2 [\wp'(u) + 2]^{1/3} \operatorname{sgn}\varphi + \varphi^2 \wp(u)\} \qquad (0 \leqslant N\psi < {}^2/_3\omega)$$

Upstream of the shock wave the equation of the sonic line is

$$\varphi = 2^{1/3}\varphi_c [2 - \wp'(u)]^{-1/3} \qquad (\alpha_1 \leqslant u < \alpha_2,\ \varphi_c > 0) \tag{2.13}$$

and intersecfs the sonic boundary at point C (where $\varphi = \varphi_c > 0$). Downstream of the shock wave a flow corresponding to solution (2.12) is throughout subsonic. There are two lines $\theta = 0$, one upstream of the shock

$$\varphi = 2^{-1/6}\sqrt{3}\,\varphi_c |2 - \wp'|^{1/6} (-\wp')^{-1/2} \qquad (\alpha_1 \leqslant u < \omega,\ 0 \leqslant N\psi < {}^1/_3\omega)$$

the other downstream of it

$$\varphi = -2^{-1/6}\sqrt{3}\varphi_c |2 - \wp'|^{1/6} (\wp')^{-1/2} \qquad (\omega < u < \alpha_2,\ {}^1/_3\omega < N\psi < {}^2/_3\omega)$$

The pattern of behavior of solution (2.12) in the $\varphi\psi$-plane is shown on Fig. 2. Transposition onto the stream plane is carried out with the aid of Formulas (1.7). The gas flow pattern is shown on Fig. 3, where streamline $N\psi = c$, ${}^1/_3\ \omega < c < {}^2/_3\ \omega$ is assumed to be a solid wall. In the neighborhood of point A the stream boundary is expressed by Eq.

$$y = (N / B_0)^3\ a_* \varphi_c{}^2 x^2 \ \operatorname{sgn}\ x$$

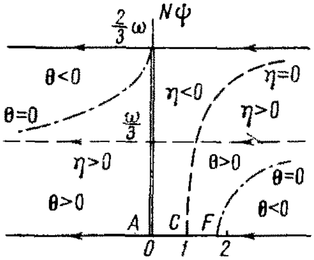

Fig. 2

The appearance of shock wave AB in the stream on Fig. 3 is due to the absence of profile convexity in the supersonic zone (see, e.g., [12 and 13] in which flows in the local supersonic zone along the profile are analyzed).

We shall investigate solution (2.12) in the neighborhood of point C, which is the point of intersection of sonic line CD with the sonic boundary FA. With this in view we expand (2.12) into a series in powers of $u - \alpha_1 = \overline{N\psi}$ and $\varphi - \varphi_c = \overline{\varphi}$, taking into account that at point C

$$\psi = 0, \quad \bar\theta = 0 \qquad (\theta = \theta - \theta_c,$$
$$\theta_c = {}^4/_3\ (N / B_0)^3 \varphi^3{}_c)$$

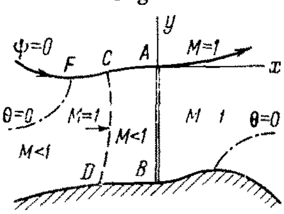

Fig. 3

As for $u \approx \alpha_1$

$$\wp(u) = -2\ (u - \alpha_1) + 2\ (u - \alpha_1)^4 - {}^8/_7\ (u - \alpha_1)^7 + \ldots$$

then retaining in the expansion of (2.12) the dominant terms only, we obtain the following solution of system (1.1) in the intersection neighborhood of sonic lines [9]:

$$\theta = -a\varphi^2 + {}^4/_3\ B_0{}^3 a^2 \varphi\ \psi^3 - {}^1/_9\ B^6{}_0 a^3 \psi^6$$
$$\eta = 2B_0 a\,\varphi\psi - {}^1/_3\ B^4{}_0 a^2 \psi^4 \tag{2.14}$$
$$a = 2\ (N / B_0)^3\, \varphi_c$$

where dashes over φ and θ have been omitted.

Thus the nonself-similar solution (2.12) of system (1.1) in the neighborhood of the sonic stream boundary CA (Fig. 1) defines the transonic flow of a perfect gas past a profile, including a shock wave.

If in (2.12) parameter $\varphi_c \to 0$, then sonic line CD (Fig. 3) approaches shock AB the intensity of which decreases, and at the limit when $\varphi_c = 0$ this solution becomes the Ovsiannikov's solution [4] which describes a subsonic flow with a normal sonic line.

3. Motion of gas in a Laval nozzle with a shock wave in its discharge part. We can derive in a similar manner a nonself-similar solution of system (1.1) for the case of a transonic flow of gas in a symmetric Laval nozzle in the presence of a weak shock wave AB in its discharge part (Fig. 4). The shock in the nozzle axis proximity may be considered to be normal.

Let $\psi = 0$ along the nozzle axis, and $\psi > 0$ below it. By virtue of the parity of function $z_0(\psi) \geqslant 0$ and $z_1(\psi)$ which define a symmetric flow of gas we have from (1.4) and (1.6)

$$z'_0(0) = 0, \qquad z'_1(0) = 0 \tag{3.1}$$

From the condition of intersection of all streamlines with the sonic line upstream of the shock wave, we derive that in (1.4) $z_1(\psi) < 0$. Integrating (1.5), and taking into account (3.1), we find that

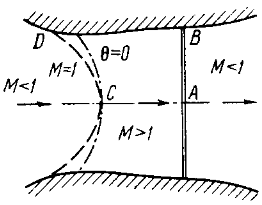

Fig. 4

$$(z'_1)^2 = 4\,B_0^3\,z_1^3 - 4\,B_0^3\,[z_1(0)]^3$$

Function $z_1(\psi)$ may, therefore, be expressed in terms of function $\wp(u) = \wp(u;\,0,\,-4)$, which is negative when $\alpha_1 < u < \alpha_2$ (see Section 2)

$$z_1(\psi) = (N^2/B_0^3)\,\wp(u), \qquad z_1'(\psi) = (N/B_0)^3\,\wp'(u)$$

$$(u = N\psi + \omega,\ N > 0,\ \alpha_1 < u < \alpha_2) \tag{3.2}$$

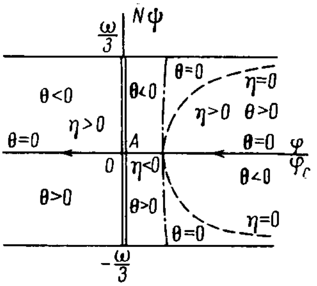

Fig. 5

The even solution $z_0(\psi) = R_0(u) \geqslant 0$ $(u = N\psi + \omega)$ satisfying Eq. (2.7) is [5]

$$z_0(\psi) = R_0(u) = 2^{-4/3}(N^2/B_0^3)\,\varphi_c^2\,\{[\wp'(u) + 2]^{1/3} + [2 - \wp'(u)]^{1/3}\}$$

$$z_0'(\psi) = NR_0'(u) = 2^{-5/3}(N/B_0)^3\,\varphi_c^2\,\{[2 - \wp'(u)]^{2/3} - [2 + \wp'(u)]^{2/3}\} \tag{3.3}$$

$$(\alpha_1 < u < \alpha_2) \qquad (u = N\psi + \omega)$$

The constant φ_c in (3.3) will be defined later.

The solution of system (1.1) corresponding to a transonic flow in a Laval nozzle in the presence of a shock wave is of the form

$$\theta = (N/B_0)^3\,\{2^{-5/3}\varphi_c^2\,[(2 - \wp')^{2/3} - (2 + \wp')^{2/3}]\,|\varphi| + {}^1/_3\varphi^3\wp'\}$$
$$\eta = -(N/B_0)^2\,\{2^{-4/3}\varphi_c^2\,[(2 + \wp')^{1/3} + (2 - \wp')^{1/3}]\,\mathrm{sgn}\,\varphi + \varphi^2\wp\} \tag{3.4}$$
$$(u = N\psi + \omega) \qquad (\,|N\psi| < {}^1/_3\omega)$$

This solution (up to the straight line *AB* on Fig. 4) was derived in [5], in this exercise it is, however, extended beyond line *AB* as a continuous and symmetric one relative to that line. In this case it would be unrealistic on physical grounds to expect a continuous compression of the supersonic stream in the second constriction of the nozzle.

If *AB* is assumed to be a shock wave, and the solution extended beyond it, then the flow in a single-throated nozzle (Fig. 4) with walls having inflection points in its discharge part would be obtained.

The pattern of behavior of solution (3.4) in the $\varphi\psi$-plane is plotted on Fig. 5.

The sonic line equation upstream of the shock wave is

$$\varphi = 2^{-2/3}\varphi_c\,[(2 + \wp')^{1/3} + (2 - \wp')^{1/3}]^{1/2}\,(-\wp)^{-1/2} \tag{3.5}$$

$$(\alpha_1 < u < \alpha_2,\ \ u = N\psi + \omega)$$

and it intersects the nozzle axis at center *C* (where $\varphi = \varphi_c > 0$). There is also a line $\theta = 0$ upstream of the shock wave which passes through *C*.

For $\varphi_c = 0$ solution (3.4) becomes a solution defining a symmetric subsonic flow with a normal sonic line [4].

If solution (3.4) is expanded in the neighborhood of point *C* (where $\psi = 0$, $\varphi = \varphi_c > 0$, $\eta = \theta = 0$) into series in powers of $u - \omega = N\psi$ and $\bar{\varphi} = \varphi - \varphi_c$, and dominant terms only retained in the latter, then, taking into account that

$$\wp(u) = -1 + 3\,(u - \omega)^2 - 3\,(u - \omega)^4 + \ldots \qquad \text{for } u \approx \omega$$

we obtain the solution of (1.1) derived by Falkovich [2] which defines a transonic flow in the neighborhood of the nozzle center *C*

$$\theta = B_0\,a^2\,\varphi\,\psi - {}^1/_6 B_0^3 a^3 \psi^3, \qquad \eta = a\varphi - {}^1/_2 B_0^2 a^2 \psi^2$$
$$a = 2\,(N\,/\,B_0)^2\varphi_c \tag{3.6}$$

in which the dash over $\bar{\varphi}$ has been omitted.

Solution (3.4) defines a transonic flow near the Laval nozzle axis, with a shock wave in its discharge part. A similar viscous flow in a Laval nozzle in the presence of a shock wave was recently investigated in [14]. When the intensity of a curved shock wave at the nozzle axis is zero, then the shock origin is at the nozzle center, in the vicinity of which the flow can be analyzed with aid of self-similar solutions of transonic flows [15].

Solutions of the Tricomi equation $\eta\psi_{\theta\theta} + \psi_{\eta\eta} = 0$ in the $\theta\eta$-plane may be derived for corresponding flows by eliminating potential φ from (2.12) and (3.4).

Other transonic flows with shock waves such as, for example, the flow of a supersonic stream with a shock wave and a free boundary past a rigid plate can be analyzed with the aid of solution (2.11).

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